ON A PROPERTY OF THE FIRST APPROXIMATION SYSTEM

(OB ODNOM SVOISTVE SISTEMY PERVOGO PRIBLIZHENIIA)

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G. K. POZHARITSKII
(Moscow)

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Considered here are properties of such solutions, of a system of linear equations with constant coefficients, for which the derivative of some known quadratic form of variables, taken with respect to these equations, is a form of constant sign. It appears that under certain easily verifiable limitations imposed on the form and its derivative, a conclusion may be reached concerning the instability, or the asymptotic stability, with respect to the part of variables for that motion which possesses a given linear system as its system of first approximation for the equations of perturbed motion.

Theorem 1. If the quadratic form W pertaining to the equation

$$\frac{dx_s}{dt} = p_{s1}x_1 + \dots + p_{sn}x_n \qquad (p_{si} = \text{const}) \tag{1}$$

has a constantly negative derivative dW/dt, and the ranks of the forms W and dW/dt are equal and W may take on negative values, then the equations (1) will have a negative characteristic number.

Remark. The rank of the form is defined as by E. Cartan [2]. It is the smallest number of linearly independent forms relating to $x_1, \dots x_n$, by which the form W can be expressed.

Let us assume that the aforementioned rank equals $p < n$ and $v_1 \dots v_p$ are the smallest number of linear forms through which W can be expressed.

Then we can write

$$\frac{dW}{dt} = \sum_{i=1}^{p} \frac{\partial W}{\partial v_i} \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} (p_j, x_1, + \dots + p_{jn}x_n) \tag{2}$$

Let $w_1, \dots w_p$ be the smallest number of forms by means of which dW/dt is expressed as

$$\frac{dW}{dt} = -(w_1^2 + \dots w_p^2) \tag{3}$$

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It is known that these forms $w_1, \ldots, w_p$ exist, since dW/dt is assumed to be constantly negative.

It follows from equation (2) that $dW/dt \equiv 0$, if $v_1 = \ldots v_p = 0$, and from equation (3) it is seen that $dW/dt \equiv 0$ only when $w_1 = \ldots v_p = 0$.

Consequently, from the system of equations $v_1 = \ldots = 0$ the system $w_1 = \ldots = w_p$ is derived, and this can occur only when the forms $w_1 \ldots, w_p$ are linear combinations of the forms $v_1 \ldots v_p$.

This means that dW/dt can be expressed by $v_1, \ldots v_p$, whereby

$$\frac{dW}{dt} = \sum_{ij=1}^{p} \beta_{ij} v_i v_j \tag{4}$$

expressed by $v_1, \ldots v_p$, will be a negative definite function of the variables $v_1, \ldots v_p$.

It is not difficult to prove that there exists such a constant $\beta > 0$ which will satisfy the inequality

$$dW / dt < \beta W$$

and also the inequality

$$W \leqslant W_0 e^{\beta(t-t_0)}$$

If W_0 can be made negative, then the basic theorem pertaining to the characteristic number of the sum and the derivative confirms that the equations (1) have a negative characteristic number.

The proof of this theorem may be repeated almost without modification for the following theorem.

Theorem 2. If under the conditions of the aforementioned theorem the quadratic form W is constantly positive, the motion will be asymptotically stable with respect to $v_1, \ldots v_p$.

The presented propositions permit an estimate of the behavior of the solutions of system (1) immediately after we convince ourselves that dW/dt is constantly negative, and the minimum minors, different from zero, of the two linear forms

$$\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n}; \quad \frac{\partial}{\partial x_1}\left(\frac{dW}{dt}\right), \ldots, \frac{\partial}{\partial x_n}\left(\frac{dW}{dt}\right)$$

are of the same order, because, as proved by E. Cartan, this order is precisely the rank of both forms.

Remark. The established result permits us to show that dW/dt will be, according to the terminology of N.G. Chetaev [3], definitely negative in the region $W < 0$. If W can attain negative values, then the functions W and dW/dt satisfy the Chetaev instability theorem.

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